

A limitation of the Nakajima-Zwanzig projection method¹

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Abstract. There is current interest in dynamical description of different decompositions of a quantum system into subsystems. We investigate usefulness of the Nakajima-Zwanzig projection method in this context. Particularly, we are interested in simultaneous description of dynamics of open systems pertaining to different system-environment splits (decompositions). We find that the Nakajima-Zwanzig and related projection methods are system-environment split specific, that every system-environment split requires specific projector, and that projector adapted to a split neither provides information about nor commute with a projector adapted to an alternative system-environment split. Our findings refer to finite- and infinite-dimensional systems and to arbitrary kinds of system-environment splitting. These findings are a direct consequence of the recently established quantum correlations relativity. We emphasize the subtlety and delicacy required of the task of simultaneously describing the dynamics of alternate system-environment splits.

PACS numbers: 03.65.Yz, 03.67.-a, 03.65.Ud

1. Introduction

There is current interest in dynamics of different decompositions of a composite quantum system into subsystems^[1–13]. While physical motivations, methods and technical details are diverse, there is a common core of the task that can be investigated on the sufficiently general background. Typically, a C system is decomposed as $C = S + E$. The point is that there are many possible such decompositions, $S + E$, $S' + E'$, etc., and one is interested e.g. in (a) dynamics of some subsystems, e.g. S , S' , as well is in (b) amount and dynamics of non-classical correlations present in different decompositions.

Within the decoherence theory,^[14,15,16,17] the task is foundational. There is not *a priori* privileged system-environment split (decomposition)^[6,7,14,15,16,17]. Quantum decoherence is typically studied starting from a fairly unprincipled

¹Project financially supported by Ministry of Science Serbia contract no 171028 and for MD also partially supported by the ICTP - SEENET-MTP grant PRJ-09 (Cosmology and Strings) within the framework of the SEENET-MTP Network.

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choice of system-environment split. In this sense, decoherence is often considered as an approximate process^[15,17]. In practice, and particularly for the "macroscopic" world, division of the total system into "(open) system" and "environment" is "imperfect"^[15]. In the quantum optics context,^[4] the task tackles validity of certain kind of master equations. In quantum electrodynamics, the task points out important implications for the ontology of noncovariant canonical QED due to the gauge freedom^[5]. In quantum information science, one can show that amount of quantum (non-classical) correlations is not merely a feature of a composite system, or of the system's state, but is a feature of the composite system's split into subsystems^[8]. The task equally refers to the system's split into "virtual subsystems" with the question of existence of the preferable split (structure) of the composite system^[7,9]. Different partitions [not necessarily bipartitions] into subsystems, where some subsystems may serve as the open system and the rest as the environment, is perhaps the main model considered in the quantum phase transitions field^[10] (and the references therein). Tensor products of an algebra of operators or observables of a composite system is the main formal framework for the task regarding the finite-dimensional systems^[1,2,3,10,11]. In quantum information practice, one can hope to be able to recognize "the preferable behavior of quantum correlations which allows a given quantum system to be more flexible in applications."^[9] So, the task is of both academic as well as of interest for different kinds of applications. A large interest in the topic regarding applications can be found e.g. in Ref. [11]. Some prospects can be found in Refs. [12,13]. For a recent review of a part of the topic see Ref. [13].

In this paper, we launch a variant of the task that is of the kind (a) described above and that is a common core for the most of the research results^[1–13]. Actually, we are interested in *simultaneous* dynamics of a pair of *open* systems formally denoted S and S' , which pertain to different decompositions (structures) of a composite system. Specific for our considerations is that we consider the task in the context of the Nakajima-Zwanzig projection method in the open quantum systems theory^[18,19].

A part of motivation for the present paper is the fact that the Nakajima-Zwanzig projection method^[20,21] is *central* to modern open quantum systems theory^[18,19]. The method provides a systematic theoretical approach to Markovian dynamics and sets a basis for a systematic (perturbative) approach to non-Markovian dynamics^[18,19]. The open systems theory provides the foundations of quantum measurement theory^[3,22], decoherence^[1,2,3], and the emergence of thermodynamic behavior^[23]. Applications of the theory of open quantum systems are found in practically all areas of physics, ranging from quantum optics^[24], quantum information^[25] and condensed-matter physics^[26] to chemical physics^[27] and spintronics^[28].

The key idea behind the Nakajima-Zwanzig projection method consists of the introduction of a certain projection operator, \mathcal{P} , which acts on the operators of the state space of the total system "system+environment" ($S + E$). If ρ is the density matrix of the total system, the projection $\mathcal{P}\rho$ (the "relevant part" of the total density matrix) serves to represent a simplified effective description through a reduced state of the total system. The complementary part (the "irrelevant part" of the total density matrix), $\mathcal{Q}\rho = (I - \mathcal{P})\rho$. For the "relevant part", $\mathcal{P}\rho(t)$, one derives closed equations of motion in the form of integro-differential equation. The open system's density matrix $\rho_S(t) = \text{tr}_E \mathcal{P}\rho(t)$ contains *all* necessary information about the open system S .

The Nakajima-Zwanzig projection method assumes a concrete, in advance chosen and fixed, system-environment split (a "structure"), $S + E$, which is uniquely defined by the associated tensor product structure (TPS) of the total system's Hilbert space, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$. Division of the composite system into "system" and "environment" is practically motivated. In principle, the projection method can equally describe arbitrary system-environment split i.e. arbitrary factorization of the total system's Hilbert state. At the time when importance of quantum correlations was not acknowledged, the Nakajima-Zwanzig method appeared to be a whole that can not and should not be improved. But the existence of non-classical correlations shed new light on versatility of the method.

Interestingly enough, we find that the projection methods are generally *unsuitable* for the task of *simultaneous* description of *open* systems S and S' . Our finding is general: it refers to the finite- as well as the infinite-dimensional systems and to all kinds of the variables transformations, which induce the tensor-product structures of the composite system's Hilbert state space. Our results are due ultimately to the recently established quantum correlations relativity^[8]. It is therefore not surprising that we are only now able to distinguish the following findings as the basis for our main result: (i) first, [not very surprisingly], every system-environment split requires a specific projector; (ii) the projection-based information about the S system is in general not sufficient for drawing information about the S' system at the *same* time; (iii) one cannot construct mutually compatible (commuting) projectors that pertain to different decompositions *simultaneously*.

Our findings, that go beyond the standard thinking^[18,19] of the open quantum systems, do not present any inconsistency with the open systems theory or with the foundations of the Nakajima-Zwanzig method. Rather, our findings point out that the Nakajima-Zwanzig projection method has a *limitation*, i.e. is *not suitable* for the above posed task. Finally, we emphasize subtlety and delicacy of simultaneous dynamical description of the open sys-

tems pertaining to different system-environment splits.

2. Simultaneous dynamics of the structures

A composite system C can be differently decomposed into "system+environment", $S + E$ and $S' + E'$. We wonder if, within the projection method, the unitary dynamics of C can provide *simultaneous* (i.e. in the same time interval) reduced dynamics for both open systems, S and S' . While description of the different structures at the same time is basic—notice the simultaneous redefinition of both "system" and "environment"—it can also have some *practical* motivations. E.g. we can wonder if the S' system can be more easily accessible in a laboratory than the S system. This can provide the more convenient recipes for manipulating the C 's degrees of freedom. Or we may be interested in dynamics of the S' system, which is not directly accessible in a laboratory.

Quantum mechanics is insensitive to different structures (decompositions in to parts) of a composite system C . That is, quantum mechanics equally treats the different structures of C . The von Neumann-Liouville equation ($\hbar = 1$):

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] \quad (1)$$

where H is the total system's Hamiltonian acting on the total system's Hilbert state-space \mathcal{H} , equally applies to every decomposition (structure) of C .

Consider a pair of structures, $S + E$ and $S' + E'$; $S + E = C = S' + E'$. This provides the different tensor-product-structures for the total Hilbert state space, $\mathcal{H}_S \otimes \mathcal{H}_E = \mathcal{H} = \mathcal{H}_{S'} \otimes \mathcal{H}_{E'}$. Of course, the total system's Hamiltonian and state, as well as reduced state of any subsystem (obtained by the proper tracing out) are unique in every instant of time. The different forms of the total Hamiltonian

$$H^{(SE)} \equiv H_S + H_E + H_{SE} = H = H_{S'} + H_{E'} + H_{S'E'} \equiv H^{(S'E')}, \quad (2)$$

where the double subscripts distinguish the interaction terms. Then eqs. (1) and (2) provide simultaneous description of the reduced dynamics for both open systems, S and S' :

$$\frac{d\rho_i(t)}{dt} = -i \text{tr}_j[H, \rho(t)], \quad i = S, S', \quad j = E, E'; \quad (3)$$

for the $i = S$, the Hamiltonian H takes the form $H^{(SE)}$, while for the $i = S'$, the Hamiltonian takes the form $H^{(S'E')}$.

The technical difficulties in solving equations eq.(3) have historically led to the development of the different methods, notably to the Nakajima-Zwanzig projection method^[20,21], which introduces a projection operator \mathcal{P} and its complementary projection operator, $\mathcal{Q} = I - \mathcal{P}$. The projection $\mathcal{P}\rho(t)$ is *required* to contain *all* necessary information about the open system S :

$$\rho_S(t) = \text{tr}_E \rho(t) = \text{tr}_E \mathcal{P}\rho(t) \Leftrightarrow \text{tr}_E \mathcal{Q}\rho(t) = 0, \forall t. \quad (4)$$

Then the task is to provide a closed master equation for $\rho_S(t)$, such as e.g. the generalized Nakajima-Zwanzig equation or the time-convolutionless master equation^[18,19].

The linear projections fulfilling eq.(4) can be defined^[14,23]: (i) $\mathcal{P}\rho(t) = (\text{tr}_E \rho(t)) \otimes \rho_E$ [for some $\rho_E \neq \text{tr}_S \rho$], (ii) $\mathcal{P}\rho(t) = \sum_n (\text{tr}_E P_{Sn} \rho(t)) \otimes \rho_{En}$ [with arbitrary orthogonal supports for ρ_{En}], and (iii) $\mathcal{P}\rho(t) = \sum_i (\text{tr}_E P_{Ei} \rho(t)) \otimes P_{Ei}$ [with arbitrary orthogonal projectors for the E system]; by P , we denote the projectors on the respective Hilbert state (factor) spaces. The physical context fixes the choice of the projection—e.g. by an assumption about the initial state. In this paper we stick to the projection (i), which is by far of the largest interest in foundations and applications of the open systems theory.

To illustrate our point, consider the standard quantum teleportation setup [25] for three qubits, $C = 1 + 2 + 3$. The two bipartite structures, $S + E \equiv 1 + (2 + 3)$ and $S' + E' \equiv (1 + 2) + 3$. Then the quantum state $|\Psi\rangle$ of C can be written as [25]

$$|u\rangle_S \otimes |\phi\rangle_E = |\Psi\rangle = \sum_{i=1}^4 \frac{1}{2} |i\rangle_{S'} \otimes |i\rangle_{E'}. \quad (5)$$

The projection (i) gives $\mathcal{P}'\rho = (1/4) \sum_i |i\rangle_{S'} \langle i| \otimes \rho_{E'}$ for the $S' + E'$ structure that is a mixed state fulfilling the condition eq.(4). Application of the same procedure for the $S + E$ structure, i.e. the projection (i), gives the equality $\mathcal{P}\rho = \rho$ for the $S + E$ structure also satisfying eq.(4). The projector \mathcal{P}' provides a mixed state for the total system. However, the \mathcal{P} projector provides a pure state for the total system. So, the two projection operators, \mathcal{P} and \mathcal{P}' , cannot equal to each other. Moreover, as they provide different states e.g. for the 1 subsystem in an instant in time, the two projectors *exclude* each other. The same conclusion applies to the hydrogen atom differently structured either as "electron+proton ($e + p$)" or as the "center of mass + relative position ($CM + R$)". The atom's (instantaneous) state $|\Phi\rangle$ ^[17,29]

$$\sum_l c_l |l\rangle_e \otimes |l\rangle_p = |\Phi\rangle = |\chi\rangle_{CM} \otimes |nlm_l m_s\rangle_R; \quad (6)$$

in eq.(6), the quantum numbers n, l, m_l, m_s are the standard quantum numbers known from the quantum theory of the hydrogen atom.

The equalities eqs.(5) and (6) are instances of a quantum mechanical rule of "entanglement relativity" (ER),^[1,2,3,6,7,8,11,12,13] (and the references therein), which has recently been extended to relativity of the more general non-classical (quantum) correlations quantified by "discord"^[8]—quantum correlations relativity (QCR)^[8,13]. Quantum correlations relativity emphasizes: a transformation of (a change in) the degrees-of-freedom typically effects in a *change* in correlations present in the composite system C . For instance (for arbitrary instant in time), a tensor product state for one structure is endowed by non-classical correlations for the alternate structure—e.g. for a mixed state ρ ($\rho^2 \neq \rho, \text{tr} \rho = 1$)^[8,13]

$$\rho_S \otimes \rho_E = \rho = \sum_i \lambda_i \rho_{S'i} \otimes \rho_{E'i}, \quad \sum_i \lambda_i = 1, \quad (7)$$

where in general the density matrices for the $S' + E'$ structure are neither linearly dependent nor commuting. Exceptions to QCR are not ruled out. However, such exceptions become irrelevant in the dynamical analysis to be presented below.

If eq.(7) is a consequence of the projection (i) (when the ρ in eq.(7) should be exchanged by $\mathcal{P}\rho$) then it contains all necessary information about the open system S , i.e. the requirement eq.(4) is fulfilled. However, the rhs of eq.(7) is in general not of any type of the above distinguished projections (i)-(iii). Likewise for eqs.(5) and (6), the projection $\sum_i \lambda_i \rho_{S'i} \otimes \rho_{E'i}$ does not in general encapsulate all necessary information about the open system S' . Rather, the "irrelevant part", $\mathcal{Q}\rho$, can be expected to bring some information about the open system S' .

Given eq.(4) is fulfilled, eq.(7) (with the $\mathcal{P}\rho$ instead of ρ) implies:

$$\text{tr}_E \mathcal{Q}\rho(t) = \text{tr}_E(\rho(t) - \mathcal{P}\rho(t)) = \text{tr}_E(\rho(t) - \rho_S(t) \otimes \rho_E) = 0, \forall t. \quad (8)$$

The analog condition regarding the $S' + E'$ structure, for the same time instant:

$$\text{tr}_{E'} \mathcal{Q}\rho(t) = \text{tr}_{E'}(\rho(t) - \rho_S(t) \otimes \rho_E) = 0, \forall t. \quad (9)$$

Then both $\text{tr}_E \mathcal{P}\rho(t) = \rho_S(t)$ and $\text{tr}_{E'} \mathcal{P}\rho(t) = \rho_{S'}(t)$ and one can write sumultaneos (for the same time interval) master equations for the S and S' systems, with the constraints coming from eq.(7).

However, eq.(9) is *not* fulfilled. More precisely:

Lemma 1. For the most part of the composite system's dynamics, validity of eq.(8) implies nonvalidity of eq.(9), and *vice versa*.

Proof: Given eq.(8), i.e. $tr_E \mathcal{Q}\rho(t) = 0, \forall t$, we investigate the conditions that should be fulfilled in order for eq.(9), i.e. $tr_{E'} \mathcal{Q}\rho(t) = 0, \forall t$, to be fulfilled. The \mathcal{Q} projector refers to the $S + E$, not to the $S' + E'$ structure. Therefore, in order to calculate $tr_{E'} \mathcal{Q}\rho(t)$, we use ER. We refer to the projection (i) in an instant of time:

$$\mathcal{P}\rho = (tr_E \rho) \otimes \rho_E. \quad (10)$$

A) Pure state $\rho = |\Psi\rangle\langle\Psi|$, while, due to eq.(8), $tr_E \mathcal{Q}|\Psi\rangle\langle\Psi| = 0$.

We consider the pure state presented in its (not necessarily unique) Schmidt form

$$|\Psi\rangle = \sum_i c_i |i\rangle_S |i\rangle_E, \quad (11)$$

where $\rho_S = tr_E |\Psi\rangle\langle\Psi| = \sum_i p_i |i\rangle_S \langle i|$, $p_i = |c_i|^2$ and for arbitrary $\rho_E \neq tr_S |\Psi\rangle\langle\Psi|$. Given $\rho_E = \sum_\alpha \pi_\alpha |\alpha\rangle_E \langle\alpha|$, we decompose $|\Psi\rangle$ as:

$$|\Psi\rangle = \sum_{i,\alpha} c_i C_{i\alpha} |i\rangle_S |\alpha\rangle_E, \quad (12)$$

with the constraints:

$$\sum_i |c_i|^2 = 1 = \sum_\alpha \pi_\alpha, \sum_\alpha |C_{i\alpha}|^2 = 1, \forall i, \quad (13)$$

Then

$$\mathcal{Q}|\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi| - \sum_{i,\alpha} p_i \pi_\alpha |i\rangle_S \langle i| \otimes |\alpha\rangle_E \langle\alpha|. \quad (14)$$

We use ER:

$$|i\rangle_S |\alpha\rangle_E = \sum_{m,n} D_{mn}^{i\alpha} |m\rangle_{S'} |n\rangle_{E'} \quad (15)$$

with the constraints:

$$\sum_{m,n} D_{mn}^{i\alpha} D_{mn}^{i'\alpha'*} = \delta_{ii'} \delta_{\alpha\alpha'}. \quad (16)$$

With the use of eqs.(12) and (15), eq.(14) reads:

$$\sum_{m,m',n,n'} \left[\sum_{i,i',\alpha,\alpha'} c_i C_{i\alpha} c_{i'}^* C_{i'\alpha'}^* D_{mn}^{i\alpha} D_{m'n'}^{i'\alpha'*} - \sum_{i,\alpha} p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n'}^{i\alpha*} \right] |m\rangle_{S'} \langle m'| \otimes |n\rangle_{E'} \langle n'|. \quad (17)$$

After tracing out, $tr_{E'}$:

$$\sum_{m,m'} \{ \sum_{i,\alpha,n} \sum_{i',\alpha'} c_i C_{i\alpha} c_{i'}^* C_{i'\alpha'}^* D_{mn}^{i\alpha} D_{m'n}^{i'\alpha'*} - p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n}^{i\alpha*} \} |m\rangle_{S'} \langle m'| \quad (18)$$

Hence

$$tr_{E'} \mathcal{Q} |\Psi\rangle \langle \Psi| = 0 \Leftrightarrow \sum_{i,\alpha,n} [\sum_{i',\alpha'} c_i C_{i\alpha} c_{i'}^* C_{i'\alpha'}^* D_{mn}^{i\alpha} D_{m'n}^{i'\alpha'*} - p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n}^{i\alpha*}] = 0, \forall m, m'. \quad (19)$$

Introducing notation, $\Lambda_n^m \equiv \sum_{i,\alpha} c_i C_{i\alpha} D_{mn}^{i\alpha}$, one obtains:

$$tr_{E'} \mathcal{Q} |\Psi\rangle \langle \Psi| = 0 \Leftrightarrow A_{mm'} \equiv \sum_n [\Lambda_n^m \Lambda_n^{m'*} - \sum_{i,\alpha} p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n}^{i\alpha*}] = 0, \forall m, m'. \quad (20)$$

Notice:

$$\sum_m A_{mm} = 0. \quad (21)$$

which is equivalent to $tr \mathcal{Q} |\Psi\rangle \langle \Psi| = 0$, see eq.(14).

B) Mixed (e.g. non-entangled) state.

$$\rho = \sum_i \lambda_i \rho_{Si} \rho_{Ei}, \quad \rho_{Si} = \sum_m p_{im} |\chi_{im}\rangle_S \langle \chi_{im}|, \rho_{Ei} = \sum_n \pi_{in} |\phi_{in}\rangle_E \langle \phi_{in}|, \quad (22)$$

In eq.(22), having in mind eq.(10), $tr_E \mathcal{Q} \rho = 0$, while $tr_E \rho = \sum_p \kappa_p |\varphi_p\rangle_S \langle \varphi_p|$, and $\rho_E = \sum_q \omega_q |\psi_q\rangle_E \langle \psi_q| \neq tr_S \rho$.

Constraints:

$$\sum_i \lambda_i = 1 = \sum_p \kappa_p = \sum_q \omega_q, \quad \sum_m p_{im} = 1 = \sum_n \pi_{in}, \forall i. \quad (23)$$

Now we make use of ER and, for comparison, we use the same basis $\{|a\rangle_{S'} |b\rangle_{E'}\}$

$$|\chi_{im}\rangle_S |\phi_{in}\rangle_E = \sum_{a,b} C_{ab}^{imn} |a\rangle_{S'} |b\rangle_{E'}, |\varphi_p\rangle_S |\psi_q\rangle_E = \sum_{a,b} D_{ab}^{pq} |a\rangle_{S'} |b\rangle_E. \quad (24)$$

Constraints:

$$\sum_{a,b} C_{ab}^{imn} C_{ab}^{im'n'^*} = \delta_{mm'} \delta_{nn'}, \quad \sum_{a,b} D_{ab}^{pq} D_{ab}^{p'q'^*} = \delta_{pp'} \delta_{qq'}. \quad (25)$$

So

$$\mathcal{Q}\rho = \rho - (tr_E \rho) \otimes \rho_E = \sum_{a,a',b,b'} \left\{ \sum_{i,m,n} \lambda_i p_{im} \pi_{in} C_{ab}^{imn} C_{a'b'}^{imn*} - \sum_{p,q} \kappa_p \omega_q D_{ab}^{pq} D_{a'b'}^{pq*} \right\} |a\rangle_{S'} \langle a'| \otimes |b\rangle_{E'} \langle b'|. \quad (26)$$

Hence

$$tr_{E'} \mathcal{Q}\rho = 0 \Leftrightarrow \Lambda_{aa'} \equiv \sum_{i,m,n,b} \lambda_i p_{im} \pi_{in} C_{ab}^{imn} C_{a'b}^{imn*} - \sum_{p,q,b} \kappa_p \omega_q D_{ab}^{pq} D_{a'b}^{pq*} = 0, \forall a, a'. \quad (27)$$

Again, for $a = a'$:

$$\sum_a \Lambda_{aa} = 0, \quad (28)$$

as being equivalent with $tr \mathcal{Q}\rho = 0$, see eq.(26).

Validity of eq.(9) assumes validity of eq.(20) for pure and of eq.(27) for mixed states. Both eq.(20) and eq.(27) represent the sets of simultaneously satisfied equations. We do not claim non-existence of the particular solutions to eq.(20) and/or to eq.(27), e.g. for the finite-dimensional systems. Nevertheless, we want to emphasize that the number of states they might refer to is apparently negligible compared to the number of states for which this is not the case. For instance, already for the fixed a and a' , a small change e.g. in κ_s (while bearing eq.(23) in mind) undermines equality in eq.(27).

Quantum dynamics is continuous in time. Provided eq.(8) is fulfilled, validity of eq.(9) might refer *only* to a special set of the time instants. So we conclude: for the most part of the open S' -system's dynamics, eq.(9) is not fulfilled. By exchanging the roles of eq.(8) and eq.(9) in our analysis, we obtain the reverse conclusion, which completes the proof. Q.E.D.

Lemma 1 establishes: as long as eq.(4) (i.e. eq.(8)) is valid for every instant in time, the analogous equality

$$\rho_{S'}(t) = tr_{E'} \rho(t) = tr_{E'} \mathcal{P}\rho(t), \quad (29)$$

cannot be fulfilled for the most part of the open S' -system's dynamics, and *vice versa*. Then, as emphasized above, for the most part of the composite system's dynamics, the projection $\mathcal{Q}\rho$ ($\mathcal{Q}'\rho$) brings some information about the open system S' (S)—in *contradiction* with the basic *idea* of the Nakajima-Zwanzig projection method^[18,19,20,21].

Regarding the simultaneous projecting:

Lemma 2. The two structure-adapted projectors \mathcal{P} and \mathcal{P}' do not mutually commute for the projection (i).

Proof: The commutation condition, $[\mathcal{P}, \mathcal{P}']\rho(t) = 0, \forall t$. With the notation $\rho_P(t) \equiv \mathcal{P}\rho(t)$ and $\rho_{P'}(t) \equiv \mathcal{P}'\rho(t)$, the commutativity reads: $\mathcal{P}\rho_{P'}(t) = \mathcal{P}'\rho_P(t), \forall t$. Then $\mathcal{P}\rho_{P'}(t) = \text{tr}_{E'}\rho_{P'}(t) \otimes \rho_E = \rho_S(t) \otimes \rho_E$, while $\mathcal{P}'\rho_P(t) = \text{tr}_E\rho_P(t) \otimes \sigma_{S'}(t) \otimes \sigma_{E'}$. So, the commutativity requires the equality $\sigma_{S'}(t) \otimes \sigma_{E'} = \rho_S(t) \otimes \rho_E, \forall t$. However, quantum dynamics is continuous in time. Likewise in Proof of Lemma 1, quantum correlations relativity guarantees, that for the most of the time instants the equality will not be fulfilled. Q.E.D.

Lemma 2 establishes: for any pair of structures, $S + E$ and $S' + E'$, one *cannot* choose/construct a pair of compatible projectors pertaining to the projection (i) in the same time interval.

Thus the Nakajima-Zwanzig projection method faces a limitation. While it can be separately performed for any structure (either \mathcal{P} or \mathcal{P}'), it cannot be *simultaneously* used for a pair of structures. Once performed, the projection does not in general allow for drawing complete information about an alternative structure of the composite system—projecting is non-invertible (“irreversible”).

Our finding refers to *all projection-based methods*. In formal terms: Lemma 1 implies that $d\mathcal{P}\rho(t)/dt$ allows tracing out over only one structure of the composite system. If that structure is $S + E$, then $\text{tr}_{E'}d\mathcal{P}\rho(t)/dt \neq d\rho_{S'}(t)/dt$ [as long as $\rho_{S'}(t) = \text{tr}_{E'}\rho(t)$]. That is, eq.(4) for the $S' + E'$ structure is not fulfilled, and therefore cannot provide a projection-based master equation for the S' system. This can be seen also from the following argument, which is not restricted to the projection-based methods. Tracing out over E is dependent on, but not equal to, the tracing out over E' , and *vice versa*. This dependence follows from the fact that the S and E degrees of freedom are intertwined with the S' and E' degrees of freedom. Intuitively: “ tr_E ” (e.g. integrating over the E ’s degrees of freedom) partly encompasses both the S' and the E' degrees of freedom. On the other hand, Lemma 2 excludes simultaneous projecting, i.e. simultaneous master equations for the two structures. E.g., $d\mathcal{P}\rho(t)/dt = d\rho_S(t)/dt \otimes \rho_E$ is in conflict with $d\mathcal{P}'\rho(t)/dt = d\rho_{S'}(t)/dt \otimes \rho_{E'}$: due to eq.(7), only one of them can be correct for arbitrary instant in time.

Summarizing: if we consider simultaneous (i.e. for the same time interval) dynamics for the open systems pertaining to a pair of system-environment splits, Lemma 1 establishes that projection adapted to one structure cannot be used for deriving master equation regarding another structure, while Lemma 2 emphasizes that simultaneous projecting for the two structures is

not allowed. Hence, in order to simultaneously describe dynamics of the two open systems, S and S' , one should avoid projecting of the composite system's state.

Our findings take us back to the beginning, i.e. to eq.(3), which does not have any limitation. It seems that there is not a universal shortcut in deriving master equations regarding the alternative structures of a composite system.

3. Discussion

We are interested in the variables transformations that simultaneously redefine both the open system and its environment. The transformations include regrouping of the constituent particles—e.g. in "entanglement swapping"^[25,30], which is illustrated by eq.(5)—or the more general transformations as illustrated by eq.(6). Such transformations are examples of the more general linear canonical transformations performed on the total system "system+environment"^[6,9,10,13]. The transformations that target only the open system without altering the environment can be found in^[4,5,7].

Investigating the alternative system-environment splits goes beyond the standard, practically inspired methods in the open systems theory. This new research line is still in its infancy but is of interest for both academic as well as for applied research (see Introduction for the references).

Despite the fact that QCR can have exceptions for certain states, our findings presented by Lemma 1 and 2 do not. Even if QCR does not apply to an instant in time (i.e. to a special state of the total system), it is most likely to apply already for the next instant of time in the unitary (continuous in time) dynamics of the total system C . This general argument makes Lemma 1 and 2 universal, i.e. applicable to every Hilbert state space and every model and structure (the choice of the open systems S and S') of the total system. So Lemma 1 and Lemma 2 refer to the finite- and infinite-dimensional systems and to all kinds of the variables transformations.

Lemma 2 forbids construction of compatible projectors for a pair of the system-environment splits. So the only way to use the projection method is to have fulfilled the conditions $\rho_S(t) = \text{tr}_E \mathcal{P} \rho(t)$ and $\rho_{S'}(t) = \text{tr}_{E'} \mathcal{P} \rho(t)$ for every instant in time t . However, according to Lemma 1, these equalities (see eq.(8) and eq.(9)) cannot both be fulfilled for every instant in time—moreover, the equalities are not fulfilled for the most of the time instants.

Thus we are forced to conclude that the task of simultaneous description of the different structures reduces to eq.(3), yet with the constraint imposed by quantum correlations relativity^[8,13].

Consider the simplest case for the $S + E$ structure: the tensor product initial state, the environment E is harmonic bath of non-interacting oscillators weakly interacting with the open system S , applicability of the Born-Markov and of the rotating-wave approximation. Then from eq.(7) we can directly draw the following conclusions regarding the alternative structure $S' + E'$: (a) due to the presence of the initial correlations in eq.(7), the S' system's dynamics (described by eq.(3)) is non-Markovian, and also possibly non-completely positive^[18,19,31,32,33,34,35]; (b) the new environment E' is in general not in thermal equilibrium—in general it is in non-stationary state; (c) in general, the environment E' consists of mutually interacting particles. In addition to this, both the strength of interaction and validity of the rotating wave approximation can be at stake for the alternative $S' + E'$ structure. Thus, in general, investigating the alternative open system's dynamics is a formidable task, but see^[6,9].

4. Conclusion

Relativity, i.e. structure dependence, of quantum correlations limits application of the Nakajima-Zwanzig and of the related projection methods in investigating the system-environment splits. The projection-methods-provided information about a subsystem of a composite system in an instant of time is insufficient to acquire information about another subsystem of the same composite system in the same instant of time. This limitation of the projection methods suggests that "shortcuts" for describing the alternative system-environment-splitting dynamics may be non-reliable and delicate.

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